If magnetic sidewalls are assumed for the structure, then the functions Φ_e and Φ_h can be expressed in the form

$$\Phi_e(y)=\cos\beta_n y \qquad \Phi_h(y)=\sin\beta_n y.$$

But in the case of electric sidewalls Φ_e and Φ_h will be

$$\Phi_e(y)=\sin\beta_n y \qquad \Phi_h(y)=\cos\beta_n y$$

Appendix II

By using (12), (2), and (3) yield

$$\sum_{i=1}^{N_i}\left\{\left(1-\hat{G}^{ii}_{e12}\right)\begin{bmatrix}E^i_i\\ E^i_{i+1}\end{bmatrix}-\sum_{j=1}^{N_i}\hat{G}^{ij}_{e12}[\phi_1\phi_2]\begin{bmatrix}E^i_j\\ E^i_{j+1}\end{bmatrix}-\sum_{j=1}^{N_i}\left[\frac{c_2}{c_1}\hat{G}^{ij}_{e11}[\phi_1\phi_2]\begin{bmatrix}\partial_n E^i_j\\ \partial_n E^i_{j+1}\end{bmatrix}+\frac{a}{2c_1}\hat{G}^{ij}_{e11}[-1\,1]\begin{bmatrix}H^i_j\\ H^i_{j+1}\end{bmatrix}\right]\right\}=0$$

$$\sum_{i=1}^{N_i}\left\{\left(1-\hat{G}^{ii}_{h12}\right)[\phi_1\phi_2]\begin{bmatrix}H^i_i\\ H^i_{i+1}\end{bmatrix}-\sum_{j=1}^{N_i}\hat{G}^{ij}_{h12}[\phi_1\phi_2]\begin{bmatrix}H^i_j\\ H^i_{j+1}\end{bmatrix}-\sum_{j=1}^{N_i}\left[\frac{b_2}{b_1}\hat{G}^{ij}_{h11}[\phi_1\phi_2]\begin{bmatrix}\partial_n H^i_j\\ \partial_n H^i_{j+1}\end{bmatrix}+\frac{a}{2b_1}\hat{G}^{ij}_{h11}[-1\,1]\begin{bmatrix}E^i_j\\ E^i_{j+1}\end{bmatrix}\right]\right\}=0$$

$$\sum_{i=1}^{N_i}\left\{\left(1+\hat{G}^{ii}_{e22}\right)[\phi_1\phi_2]\begin{bmatrix}E^i_i\\ E^i_{i+1}\end{bmatrix}+\sum_{j=1}^{N_i}\hat{G}^{ij}_{e22}[\phi_1\phi_2]\begin{bmatrix}E^i_j\\ E^i_{j+1}\end{bmatrix}-\sum_{j=1}^{N_i}\hat{G}^{ij}_{e21}[\phi_1\phi_2]\begin{bmatrix}\partial_n E^i_j\\ \partial_n E^i_{j+1}\end{bmatrix}\right\}=0$$

$$\sum_{i=1}^{N_i}\left\{\left(1+\hat{G}^{ii}_{h22}\right)[\phi_1\phi_2]\begin{bmatrix}H^i_i\\ H^i_{i+1}\end{bmatrix}+\sum_{j=1}^{N_i}\hat{G}^{ij}_{h22}[\phi_1\phi_2]\begin{bmatrix}H^i_j\\ H^i_{j+1}\end{bmatrix}-\sum_{j=1}^{N_i}\hat{G}^{ij}_{h21}[\phi_1\phi_2]\begin{bmatrix}\partial_n H^i_j\\ \partial_n H^i_{j+1}\end{bmatrix}\right\}=0.$$

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Eigenvalues for Ridged and Other Waveguides Containing Corners of Angle $3\pi/2$ or 2π by the Finite Element Method

B. Schiff

Abstract—**Superelements have been developed to enable the finite element method to be used for computing eigenvalues of the Laplacian over domains containing reentrant corners of angle $3\pi/2$ or 2π. The superelements embody mesh refinement and include basis functions which emulate the singular behavior of the solution at the corner. Being compatible with linear or bilinear elements, the superelements are easily incorporated into standard finite element programs. The method has been used to compute TE and TM mode eigenvalues for ridged and other waveguides, and the results agree well with those obtained using various other methods.**

I. Introduction

Ridged and other waveguides whose cross sections contain one or more reentrant corners of angle $3\pi/2$ or 2π are frequently used in microwave devices and circuits. It is therefore

Manuscript received December 15, 1988; revised December 17, 1990.

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IEEE Log Number 9144283.

important to be able to obtain accurate eigenvalues for various modes of propagation in waveguides of this type. Eigenvalues have been computed by different authors using a variety of methods [1]–[9], and the flexibility of the finite element method would seem to make it ideally suited for this purpose. However, the standard finite element schemes yield comparatively poor results when applied to problems whose domain contains a reentrant corner, owing to the singular nature of the solution there. One method used to circumvent this difficulty is to refine the mesh locally in the region of the singularity [10], [11]. Another approach utilizes the known analytic form of the solution in the neighborhood of the singularity. The shape functions may be modified suitably over each of the elements which have a node at the corner, as in [12]; the trial function basis may be augmented by the addition of functions possessing a suitable singular behavior at the corner, as in [13] and [14]; or a singular trial function may be used over a section of a disk centered at the corner, as in [15]. The calculations to be described herein use a combination of these ideas. Instead of the usual finite elements, the region surrounding the corner is covered by a "superelement" containing a refined mesh over which the trial function is constrained to approximate the known behavior of the solution in the neighborhood of the singularity. The superelement conforms with linear and bilinear elements, and can easily be incorporated into standard finite element programs. The method is thus of very general applicability, and herein lies its chief advantage. The idea was first applied to problems in two-dimensional elasticity [16], [17], where a reentrant corner of angle 2π arises as a result of a crack in the material. Recently the method has been adapted to determine eigenvalues of the Laplacian over regions containing a reentrant corner of angle $3\pi/2$ [18]. Two superelements for such a corner were tested by utilizing them to determine eigenvalues for an L-shaped region, a problem for which highly accurate results have been obtained by a variety of methods [2]–[4], [13], [19]. In the present paper, we use these and similar superelements to determine eigenvalues for the two lowest TE or TM modes in guides of various shapes containing one or more corners of angle $3\pi/2$ or 2π. The method has been described in detail in [18], and only minor modifications are needed for an angle of 2π. The description of the method in Section II is, therefore, limited to the most essential features. The results obtained are listed and discussed in Sections III and IV, and the conclusions summarized in Section V.

II. The Computational Scheme

The TE and TM fields for the waveguide may be derived from potentials which satisfy the Helmholtz equation

$$\nabla^2\Psi(x,y,z)+k_0^2\Psi(x,y,z)=0 \qquad (1)$$

where k_0 is the propagation constant in free space. Taking the guide axis to be in the z direction, we assume that $\Psi(x,y,z)=\psi(x,y)e^{i\beta z}$ and hence obtain the equation

$$\nabla_T^2\psi(x,y)+k_T^2\psi(x,y)=0 \qquad (2)$$

where $k_T^2=k_0^2-\beta^2$ and ∇_T^2 denotes the two-dimensional Laplacian. Thus we wish to solve

$$\left(\nabla_T^2+\lambda\right)\psi(x,y)=0 \qquad \text{in } D \qquad (3)$$

with boundary conditions

$$\psi=0 \qquad \text{on } \partial D \quad \text{(TM modes)}$$

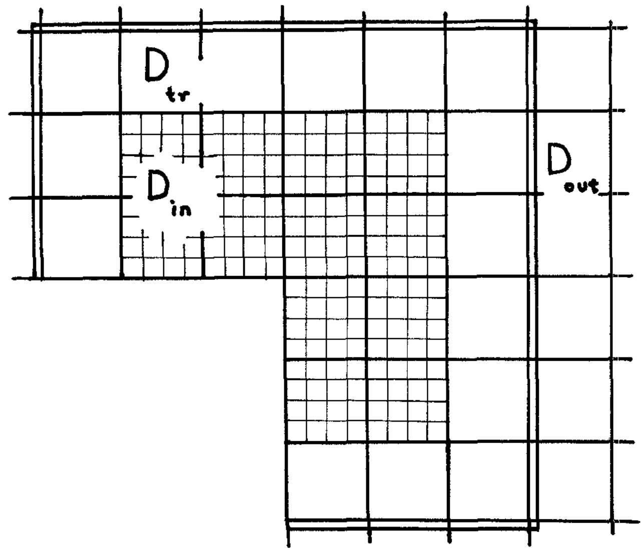

Fig. 1. L-shaped superelement embedded in external mesh D_{out}.

or

$$\frac{\partial\psi}{\partial n}=0 \qquad \text{on } \partial D \quad \text{(TE modes)} \qquad (4)$$

where the eigenvalue is $\lambda=k_T^2$.

To solve the problem by the finite element method, we use a superelement over the immediate neighborhood of each reentrant corner and cover the remainder of D with the usual elements. In the cases under discussion, the reentrant corners are all of magnitude $3\pi/2$ or 2π, and we therefore employ L-shaped or rectangular superelements. The superelement contains two regions, an interior region D_{in}, over which the original mesh is refined, and a transition region D_{tr}, which enables the trial function over D_{in} to be matched to that in D_{out}, the region outside the superelement, as shown in Fig. 1 for an angle of $3\pi/2$.

The trial function over D_{in} is designed to emulate the behavior of the exact solution in the neighborhood of the singularity. The analytic form of the series expansion for this solution is known, and the nodal values belonging to D_{in} are therefore constrained to agree with this series, suitably truncated. Over each small rectangle of D_{in}, the trial function is then taken as the bilinear interpolant to the nodal values at its four corners. The stiffness and mass matrices for D_{in} will thus be expressed with respect to the coefficients in the truncated series. The trial function in D_{tr} is taken to be piecewise bilinear, in such a way as to match up between the nodal values of D_{in} and of D_{out} on its boundary. Thus the whole superelement conforms to bilinear functions over rectangles or linear functions over triangles in D_{out}. The superelement stiffness and mass matrices will thus be with respect to the coefficients of the truncated series and the nodal values belonging to the faces abutting D_{out} as variables.

A detailed description of the construction of the superelement for an angle of $3\pi/2$ is given in [18]. In the case of a reentrant corner of angle 2π, taking polar coordinates (r,θ) with the origin at the corner, the domain will be the region $-\pi<\theta<\pi$ with a slit along $\theta=\pm\pi$. The form of the solution

TABLE I
VALUES OF k_T (IN RADIANS/UNIT LENGTH) FOR THE FIRST TWO MODES IN VARIOUS GUIDES WITH CORNERS OF ANGLE $3\pi/2$

		No. of d.o.f.	k_T (1st mode)	k_T (2nd mode)
Double L-shaped guide (TM)				
Fox *et al.*[1] and Fix *et al.*[2]			3.105	3.898
Current:	0.2×0.2 with superelement	84	3.128	3.948
Current:	0.2×0.2 without superelement	96	3.160	3.962
Current:	0.1×0.1 with superelement	329	3.114	3.914
Current:	0.1×0.1 without superelement	341	3.121	3.914
Crossed guide (TE)				
Lin[3]			7.40	21.89
Current:	with refinement and superelement	165	7.402	21.96
Current:	with refinement alone	174	7.415	21.99
Current:	without refinement	112	7.428	22.03
Symmetric double-ridged guide (TE)				
Montgomery[4]			3.650	8.041
Utsumi[5]			3.652	8.014
Israel *et al.*[6]		164	3.657	
Current:	with refinement and superelement	150	3.658	8.056
Current:	with refinement alone	159	3.667	8.062
Current:	without refinement	98	3.685	8.073
Symmetric quadruple-ridged guide (TE)				
Dasgupta and Saha[7]			1.822	2.322
Webb[8]			1.808	2.300
Current:	with refinement and superelement	304	1.811	2.308
Current:	with refinement alone	336	1.815	2.313
Current:	without refinement	125	1.858	2.372

[1]See [19].
[2]See [13].
[3]See [21].
[4]See [5].
[5]See [7].
[6]See [10].
[7]See [6].
[8]See [14].

in the neighborhood of the corner will be

$$u(r,\theta)=\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\left[c_{ij}r^{2i+j}\sin j\theta+d_{ij}r^{2i+j-1/2}\cos\left(j-\frac{1}{2}\right)\theta\right] \tag{5}$$

for boundary conditions $u=0$ over $\theta=\pm\pi$, and

$$u(r,\theta)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\gamma_{ij}r^{2i+j}\cos j\theta+\delta_{ij}r^{2i+j+1/2}\sin\left(j+\frac{1}{2}\right)\theta\right] \tag{6}$$

for boundary conditions $\partial u/\partial n=0$ over $\theta=\pm\pi$. The superelement is analogous to that for an angle of $3\pi/2$ except that it is now rectangular in shape, with a slit along $\theta=\pm\pi$. As with the former case, we include all terms with a power of r less than or equal to 4, involving a total of 12 and 15 terms respectively.

The global stiffness and mass matrices K^h and M^h for the whole region, including the contributions from the superelement matrices for each reentrant corner, are assembled in the usual manner. The global variables consist of the nodal values for each node of D_{out}, together with the series coefficients for each of the superelements. The generalized eigenvalue problem

$$K^h\vec{Q}=\lambda^h M^h\vec{Q} \tag{7}$$

is finally solved to give the eigenvalues λ_j^h, which give approximations to $(k_T)_j^2$ and the corresponding eigenvectors $\vec{Q}_j$.

III. NUMERICAL RESULTS AND DISCUSSION

In order to assess the effectiveness of the method, it was applied to various cases for which calculations have been made by other authors. We first discuss the case where the reentrant corners are of angle $3\pi/2$. The two superelements (one for each type of boundary condition in (4)) which had been constructed for the benchmark calculations [18] were employed throughout, and the results are listed in Table I, together with those of the previous authors. The number of degrees of freedom (d.o.f.) listed, i.e., the dimension of the vector $\vec{Q}$ in (7), is equal to the number of nodes belonging to D_{out} together with the total number of series coefficients. The IMSL routine EIGZS was employed to determine the eigenvalues of (7) for cases where the total number of degrees of freedom did not exceed 222. For larger problems, the core memory did not suffice, and the eigenvalues were determined one at a time using inverse iteration [20]. The matrices were stored in band form, and the various matrix operations were carried out using a local library routine, BIGMATR, which handles the disk transfers automatically, as well as performing the computations. Later computations were performed using the CDC NOSVE operating system incorporating virtual memory.

In all cases other than the first listed in Table I, the geometry of the domain required the mesh to be refined to a certain extent in the neighborhood of a corner in order to be able to insert the superelement. To illustrate the effect of this refinement and of the use of the superelement separately, we have also included in Table I (the entries labeled "with refinement alone") the results obtained in replacing the superelement by

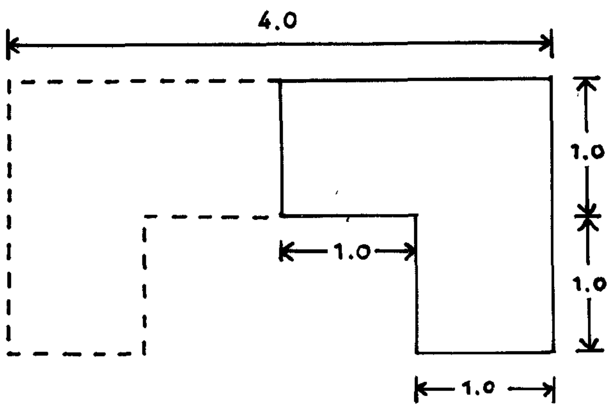

Fig. 2. Double-L-shaped single-ridged guide.

the appropriate number of bilinear elements identical in size to those used in the adjacent portion of D_{out}. It will be seen that the use of a superelement has a noticeable effect on the lowest eigenvalue, but affects the next eigenvalue to a lesser extent. The effect on the higher eigenvalues was found to be even less noticeable.

We will now describe briefly the cases considered and the methods used by other authors with whose results those of the current method are compared in Table I. The double-L-shaped single-ridged guide is shown in Fig. 2. The problem is solved over the L-shaped domain constituting half of the cross section, and detailed results have been listed for this case previously [18]. The eigenvalues obtained for the two lowest TM modes are included in Table I in order to illustrate the convergence with decreasing mesh size. In their highly accurate calculations for this case, both Fox *et al.* [19] and Fix *et al.* [13] took advantage of the symmetry of the domain and solved the problem over only a third or a half of the L-shaped region. Their calculations required significantly fewer degrees of freedom than the current method, but the methods used are not, on the other hand, of such general applicability. The results of Beaubien and Wexler [3], [4] were obtained using standard finite difference techniques, and their method is thus generally applicable. However, it makes no special provision for the presence of the singularity, and therefore entails the use of a large number of degrees of freedom (the authors themselves used up to 5000 for the L-shaped case).

TE mode eigenvalues for the symmetric double-ridged guide depicted in Fig. 3 have been obtained by Montgomery [5], Utsumi [7], and Israel and Miniowitz [10]. The problem is solved over a quarter of the cross section, a region consisting of two rectangles. Montgomery expands the solution over each rectangle into normal modes, and sets up an integral equation for the tangential component of the electric field over the interface. Application of the Ritz–Galerkin technique to the integral equation yields the eigenvalue. Utsumi, on the other hand, sets up a variational expression for the eigenvalue, using trial functions based on normal mode expansions in each rectangle. The integral over the interface again involves the tangential electric field, for which Utsumi employs trial functions which reflect the singular behavior at the reentrant corner. Both methods are, of course, limited to domains consisting of a union of rectangles, relying as they do on expansions into normal modes. The results of Lin [21] for a symmetric cross-shaped guide were obtained by applying Montgomery's method over a domain constituting a quarter of the cross section, as shown in Fig. 4. Montgomery's

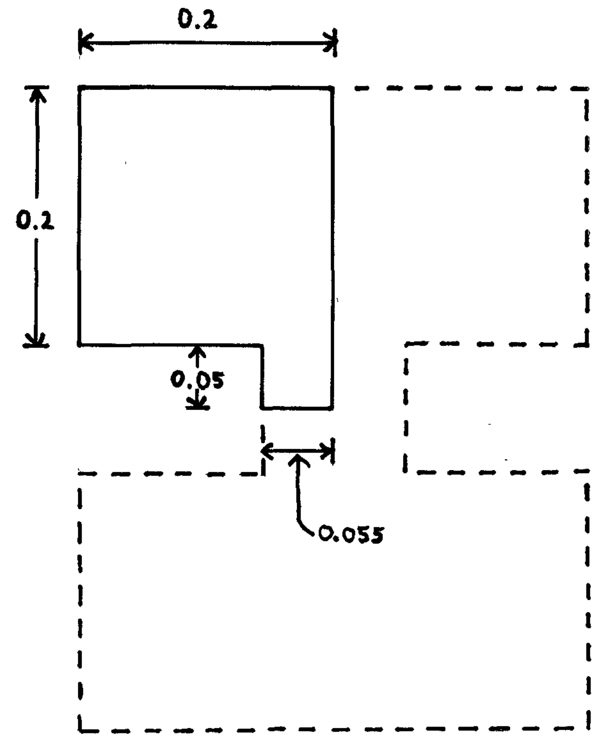

Fig. 3. Symmetric double-ridged guide.

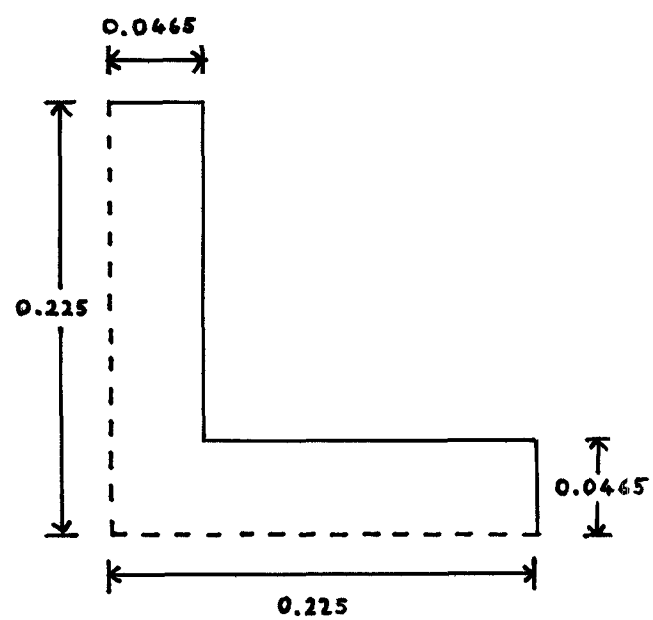

Fig. 4. Quarter of symmetrical crossed rectangular guide.

method was also applied by Dasgupta and Saha [6] to various ridged guides, and their results for a symmetric quadruple-ridged guide, whose cross section is depicted in Fig. 5, are included in Table I.

Finite element methods have also been applied to ridged waveguides by Israel and Miniowitz [10] and Webb [14]. The former authors considered the symmetric double-ridged guide of Fig. 3 using quintic Hermite elements, thus achieving C^1 continuity over the whole domain. The effect of the singularity is taken into account by refining the mesh in a graded fashion in the vicinity of the corner. The current result for the fundamen-

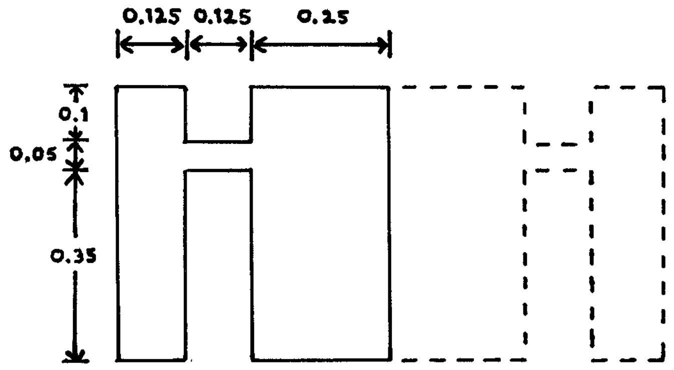

Fig. 5. Symmetric quadruple-ridged guide.

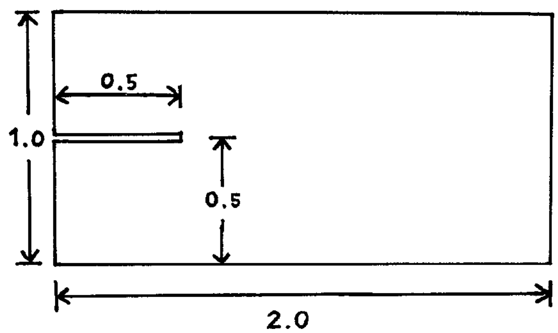

Fig. 6. Vaned rectangular guide.

TABLE II
VALUES OF k_T (IN RADIANS/UNIT LENGTH) FOR THE FIRST TWO MODES IN GUIDES WITH A CORNER OF ANGLE 2π

	No. of d.o.f.	k_T (1st mode)	k_T (2nd mode)
Unit square with slit (TM)			
Kuttler[1]		5.815	
Campbell[2]		5.799	
Current 0.1×0.1 with superelement	111	5.820	7.097
Current 0.1×0.1 without superelement	126	5.956	7.124
Current 0.05×0.05 with superelement	436	5.801	7.049
Current 0.05×0.05 without superelement	451	5.855	7.049
Vaned rectangular guide (TM)			
Swaminathan *et al.*[3]		3.677	4.928
Current with superelement	210	3.704	4.993
Current without superelement	225	3.717	5.034
Vaned rectangular guide (TE)			
Swaminathan *et al.*[3]		1.569	2.116
Current with superelement	213	1.572	2.105
Current without superelement	225	1.573	2.155

[1]See [8].
[2]See [9].
[3]See [22].

tal mode is very close to their value, as shown in Table I. Webb computes the magnetic field, imposing the zero-divergence condition via a penalty term. He treated the quadruple-ridged guide of Fig. 5 using a basis of quadratic elements augmented by functions possessing a singularity at one or other of the corners. As shown in Table I, the current results are in excellent agreement with those of Webb.

We now come to the results for a reentrant corner of angle 2π. Kuttler [8] and Campbell [9] have determined the lowest TM mode for a guide of unit square cross section ($-1\leqslant x\leqslant 1, -1\leqslant y\leqslant 1$) with a slit along the line $y=0, -1\leqslant x<0$. These authors use finite difference methods, and as with the calculations of [3] and [4] for the L-shaped case, a large number (of order 10^3 or 10^4) of degrees of freedom are needed. Swaminathan *et al.* [22] treat the case of a vaned rectangular guide, whose cross section is shown in Fig. 6. They set up an integral equation for the surface currents and apply the method of moments with piecewise-constant or piecewise-linear currents and point matching. The elements of the resulting matrix depend non-linearly on the wavenumber, and the requirement that the determinant vanish is satisfied by an iterative procedure. The results of the various calculations are listed and compared with those obtained by the current method in Table II.

IV. DISCUSSION

The advantages and disadvantages of the various methods discussed are compared in Table III. A scheme is considered to be easy to use if it can be incorporated easily into a standard finite element or finite difference program. The methods of Fix *et al.* [13] and Fox *et al.* [19] are not included in the table, as they are suitable only for very special types of domain. The results of the current method agree very well, particularly for the TE modes, with those of Webb [14], Israel and Miniowitz [10], and Swaminathan *et al.* [22]. All four schemes are suitable for polygonal domains of arbitrary shape. The current scheme, like that of Israel and Miniowitz, has the advantage of being easy to use. Furthermore, it does not require local mesh refinement near the corners. On the other hand, in contrast to the other three schemes, it is restricted to reentrant angles of $3\pi/2$ and 2π.

TABLE III
COMPARISON OF THE VARIOUS METHODS

Method	Efficient	Easy to Use	General Domain	Corners of Arbitrary Angle	Disadvantages
Finite differences[1]		√	√		Large number of degrees of freedom
Modal expansions[2]	√				Only for domains which are unions of rectangles
Surface integral equation with moment method[3]	√		√	√	Leads to nonlinear equations
Finite elements with penalty and singular functions[4]	√		√	√	Nonlocal basis (singular functions over whole domain)
Singular functions near corners, finite elements elsewhere[5]	√	√	√	√	Nonconforming trial functions
Hermite cubic finite elements[6]	√	√	√	√	Requires local mesh refinement round corners
Current	√	√	√		For angles of $3\pi/2, 2\pi$ only

[1]See [3], [4], [8], and [9].
[2]See [5]–[7], and [21].
[3]See [22].
[4]See [14].
[5]See [15].
[6]See [10].

V. CONCLUSIONS

Superelements have been developed to facilitate the determination of eigenvalues of the Laplacian over regions containing one or more reentrant corners of angle $3\pi/2$ or 2π. They have been used to determine cutoff frequencies for TE and TM modes in guides with a variety of cross sections, the results agreeing well with those obtained by other methods. The superelements are compatible with the usual linear or bilinear finite elements. They may easily be incorporated into standard finite element programs, thus enabling the latter to deal with waveguides of polygonal cross section including one or more reentrant corners of this type.

ACKNOWLEDGMENT

The author is grateful to H. Yeshurun and D. Sagin for the use of their program BIGMATR for handling out-of-core matrix problems. IMSL routines were used for the in-core case. The computations were carried out on a CDC Cyber 170/855 and later on a CDC 990 computer at the Tel Aviv University Computation Center. Helpful comments by referees are also gratefully acknowledged.

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